Higher zigzag algebras

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Motivation

- **Aim:** construct interesting periodic algebras
- **Why?** they are connected to symmetries of derived categories
- **Hope:** to get interesting relations between spherical twists

(Now) classical theory: braid group actions on triangulated categories

- **Simplest example:** various incarnations
  - (A_s)-configurations
  - Brauer tree algebras of lines
  - type A zigzag algebras
  - Ext-algebras of equivariant skyscraper sheaves
- **Why are they periodic?**
  - almost Koszul dual to type A preprojective algebras
  - type A preprojective algebras are periodic
  - (one) reason: nice AR theory of finite type hereditary algebras
- **Bite:** higher AR theory and higher preprojective algebras [Iyama, ...]

Zigzag algebras

- Fix a nice field $k = \mathbb{F} = \mathbb{C}$. Given a simple graph (no multiple edges) we construct its zigzag algebra.
- **Example:** given A_4 graph $1 \frac{\quad}{} 2 \frac{\quad}{} 3 \frac{\quad}{} 4$ we get $Z_4 = kQ/I$ where

$$Q = \; 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 3 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 4\,, \quad I = (\alpha^2, \beta^2, \alpha\beta - \beta\alpha).$$

- Radical series of indecomposable projective modules:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 1\;\,3 & 2\;\,4 & 3 \\ 1 & 2 & 3 & 4 \end{array}$$

  - Finite dimensional algebra
  - Symmetric algebra ("0-Calabi-Yau")
  - Indec projectives are spherical, i.e., $\mathrm{End}_A(P) \cong k[x]/(x^2)$
  - Maps between projectives are easy: $i \neq j \Rightarrow \dim_k \mathrm{Hom}_A(P_i, P_j) \leq 1$

Zigzag algebras

- Instead of starting with (undirected) graph, start with a quiver.
- **Example:** Bipartite type A_4: $Q = 1 \to 2 \leftarrow 3 \to 4$.
  Let $\Lambda = kQ$. Projective (left and right) modules look like:

$$
{}_\Lambda\Lambda = \begin{array}{c}1\\ \end{array} \quad \begin{array}{c}2\\ 1\ 3\end{array} \quad \begin{array}{c}3\\ \end{array} \quad \begin{array}{c}4\\ 3\end{array}; \qquad \Lambda_\Lambda = \begin{array}{c}1\\ 2\end{array} \quad \begin{array}{c}2\\ \end{array} \quad \begin{array}{c}3\\ 2\ 4\end{array} \quad \begin{array}{c}4\\ \end{array}
$$

  So injective (left and right) modules look like:

$$
{}_\Lambda\Lambda^* = \begin{array}{c}2\\ 1\end{array} \quad \begin{array}{c}\\ 2\end{array} \quad \begin{array}{c}2\ 4\\ 3\end{array} \quad \begin{array}{c}\\ 4\end{array}; \qquad {\Lambda^*}_\Lambda = \begin{array}{c}\\ 1\end{array} \quad \begin{array}{c}1\ 3\\ 2\end{array} \quad \begin{array}{c}\\ 3\end{array} \quad \begin{array}{c}3\\ 4\end{array}
$$

- Now take the *trivial extension* $\mathrm{Triv}(\Lambda) = \Lambda \oplus \Lambda^*$.
  Multiplication: $(a, f)(b, g) = (ab, fb + ag)$. Its projectives are:

$$
\begin{array}{c}1\\ 2\\ 1\end{array} \qquad \begin{array}{c}2\\ 1\ 3\\ 2\end{array} \qquad \begin{array}{c}3\\ 2\ 4\\ 3\end{array} \qquad \begin{array}{c}4\\ 3\\ 4\end{array}
$$

- We've constructed Z_4 without using generators and relations :)

Zigzag algebras

- **Example:** Linear type A_4: $Q = 1 \to 2 \to 3 \to 4$. Projectives for kQ:

$$\begin{matrix} 1 & 2 & 3 & 4 \\ & 1 & 2 & 3 \\ & & 1 & 2 \\ & & & 1 \end{matrix}$$

- This won't work:
  Loewy length is too big!
- So make it smaller in stupidest way possible: $\Gamma = kQ/\operatorname{rad}^2 kQ$.

$${}_\Gamma\Gamma = \begin{matrix} 1 & 2 & 3 & 4 \\ & 1 & 2 & 3 \end{matrix}; \qquad {}_\Gamma\Gamma^* = \begin{matrix} 2 & 3 & 4 & \\ 1 & 2 & 3 & 4 \end{matrix}$$

- Then $\mathrm{Triv}(\Gamma)$ looks like

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1\ 3 & 2\ 4 & 3 \\ 1 & 2 & 3 & 4 \end{matrix}$$

  and we've found Z_4 again :)
- Note that $\Gamma^{\mathrm{op}} = (kQ)^!$ (quadratic dual). In fact, $Z_4 = \mathrm{Triv}((kQ)^!)$.

Derived categories

- Derived category $\mathrm{D^b}(A)$ is (!) chain complexes of projective modules
$$\cdots \xrightarrow{d} P^{(3)} \xrightarrow{d} P^{(2)} \xrightarrow{d} P^{(1)} \xrightarrow{d} P^{(0)} \xrightarrow{d} 0 \to \cdots$$
eventually zero, $d^2 = 0$, modulo $(\cdots \to 0 \to P \xrightarrow{\mathrm{id}} P \to 0 \to \cdots)$.
- Write Z_s for the type A_s zigzag algebra ($s \in \mathbb{Z}_{\geq 1}$).
- For each vertex i we have a *spherical twist* $F_i : \mathrm{D^b}(Z_s) \xrightarrow{\sim} \mathrm{D^b}(Z_s)$:
$$M \quad \mapsto \quad (P_i \otimes_k \mathrm{Hom}(P, M) \xrightarrow{\mathrm{ev}} M).$$
- **Example:** $s = 2$. The projectives are:
$$P_1 = \begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \; ; \qquad P_2 = \begin{matrix} 2 \\ 1 \\ 2 \end{matrix}$$
So $F_1(P_1) = (P_1 \otimes \langle \mathrm{id}, \alpha\beta \rangle \xrightarrow{\mathrm{ev}} P_1) \cong (P_1 \to 0)$; $F_1(P_2) = (P_1 \xrightarrow{\beta} P_2)$.
- Let $\mathrm{Br}_{s+1} = \langle \sigma_1, \ldots, \sigma_s \rangle / \sim$ denote the braid group on $s+1$ strands.
- Here, σ_i denotes crossing ith strand over $(i+1)$st strand.
- **Theorem** [ST, RZ]: Br_{s+1} acts on $\mathrm{D^b}(Z_s)$ by $s_i \mapsto F_i$.

Derived categories

- **Example:** $\sigma_1\sigma_2\sigma_1 \in \mathrm{Br}_3 \twoheadrightarrow S_3 \ni w_0 = s_1s_2s_1$ acts on $\mathrm{D^b}(Z_2)$ by:

$$P_1 \xmapsto{F_1} (P_1 \otimes \langle \mathrm{id}, \alpha\beta \rangle \xrightarrow{\mathrm{ev}} P_1) \cong (P_1 \to 0)$$

$$\xmapsto{F_2} (P_2 \otimes \langle \beta \rangle \xrightarrow{\mathrm{ev}} P_1 \to 0) \cong (P_2 \to P_1 \to 0)$$

$$\xmapsto{F_1} \begin{pmatrix} P_1 \otimes \langle \alpha \rangle & \longrightarrow & P_1 \otimes \langle \mathrm{id}, \alpha\beta \rangle & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ P_2 & \longrightarrow & P_1 & \longrightarrow & 0 \end{pmatrix} \cong (P_2 \to 0 \to 0)$$

- So $w_0 = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ acts on $\mathrm{D^b}(Z_2)$ sending P_1 to $P_2[2]$.
- Compute projective resolution of the simple Z_2-module at vertex 1:

$$\begin{array}{cccc} & & 1 \to & 1 \\ & 2 \to & 2 & \\ & 1 & 1 & \\ 2 \to & 2 & & \end{array}$$

- We see that periodicity of Z_2 [BBK] corresponds to action of w_0 [RZ].

Higher zigzag algebras

- **Definition:** For Λ Koszul with gl. dim. $< \infty$, its *higher zigzag algebra* is $Z(\Lambda) = \mathrm{STriv}(\Lambda^!)$.
- STriv is a graded/"super" version of Triv which makes the theorem below work. For "type A" examples, STriv $\cong$ Triv.
- We want a connection with type A higher preprojective algebras. Let $\Lambda = \Lambda_s^d$ be Iyama's type A d-representation finite algebra.
- **Definition:** The *type A_d^s higher zigzag algebra* is $Z_s^d := Z(\Lambda_s^d)$.
- $\exists$ explicit description $Z_s^d = kQ_s^d / I_s^d$.
- Note: For $d = 2$ these appear as endomorphism algebras of "hica"s [Miemietz-Turner] and for $d \geq 2$, Z_s^d defined independently by Guo and Luo: called "d-cubic pyramid algebras".
- **Theorem** [G-Iyama] For $s \geq 3$, $Z_s^d \cong \Pi(\Lambda_s^d)^!$.
- A morphism $\Pi^! \to Z$ always exists; surjectivity follows as Λ_s^d is d-hereditary; but injectivity is proved by a dimension count using type A combinatorics. It would be nice to better understand surjectivity.

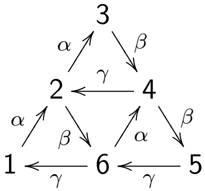
Higher zigzag algebras

Examples:

- $d = 1$: $Z_s^1 = Z_s$. (Brauer tree algebra of line with s edges)
- $s = 1$: $Z_1^d = k[x]/(x^2)$ with x in degree $d + 1$.
- $s = 2$: Z_2^d is the symmetric Nakayama algebra with $d + 1$ vertices and Loewy length $d + 2$ (Brauer tree algebra of star with $d + 1$ edges).
- $d = 2$ and $s = 3$:

$$Z_3^2 = kQ_3^2/(\alpha^2, \beta^2, \gamma^2, \alpha\beta - \beta\alpha, \alpha\gamma - \gamma\alpha, \beta\gamma - \gamma\beta)$$

$Q_3^2 =$

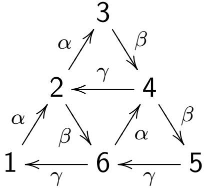


1	2	3	4	5	6
6	1 4	2	3 6	4	5 2
2	6 3	4	2 5	6	4 1
1	2	3	4	5	6

- $d = 2$ and $s = 4$: see intro to paper. $d = 3$: tetrahedra...

Higher zigzag algebras

- **Example:** $d = 2$ and $s = 3$.



1	2	3	4	5	6
6	1 4	2	3 6	4	5 2
2	6 3	4	2 5	6	4 1
1	2	3	4	5	6

- Compute projective resolutions of simples:

$1 \to 1$
$6 \to 6$
$5 \to 5\ 2 \quad 2$
$4 \quad 4\ 1 \quad 1$
$6 \quad 6$
$5 \to 5$

$2 \to 2$
$1 \oplus 4 \to 1\ 4$
$6 \to 6 \quad 3\ 6 \quad 6\ 3$
$5\ 2 \quad 2 \quad 2\ 5 \quad 2$
$4\ 1 \quad 1 \quad 4$
$6 \to 6$

- Again, we have some periodicity.

Group actions

- Again, we get spherical twist $F_i : \mathrm{D^b}(Z_s^d) \xrightarrow{\sim} \mathrm{D^b}(Z_s^d)$ at each vertex.
- $1 \rightleftarrows 2$ gives relation $F_1 F_2 F_1 \cong F_2 F_1 F_2$.
- $\begin{array}{ccc} & 2 & \\ \nearrow & & \searrow \\ 1 & \longleftarrow & 3 \end{array}$ gives relations $\begin{cases} F_1 F_2 F_3 F_1 \cong F_2 F_3 F_1 F_2 \cong F_3 F_1 F_2 F_3; \\ F_1 F_2 F_1 \cong F_2 F_1 F_2, \text{ etc.} \end{cases}$
- $\begin{array}{ccc} 2 & \longrightarrow & 3 \\ \uparrow & & \downarrow \\ 1 & \longleftarrow & 4 \end{array}$ gives relations

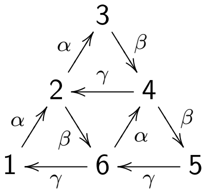
$$\begin{cases} F_1 F_2 F_3 F_4 F_1 \cong F_2 F_3 F_4 F_1 F_2 \cong F_3 F_4 F_1 F_2 F_3 \cong F_4 F_1 F_2 F_3 F_4; \\ F_1 F_2 F_3 F_1 \cong F_2 F_3 F_1 F_2, \text{ etc.}; \\ F_1 F_2 F_1 \cong F_2 F_1 F_2, \text{ etc.} \end{cases}$$

- We get large group $G_s^d = \langle \sigma_i \rangle / \sim$ acting on $\mathrm{D^b}(Z_s^d)$ by $\sigma_i \mapsto F_i$.

Group actions

- G_s^d has element w_0 playing role of "lift of longest element"
- w_0 is built from "Coxeter elements" adapted to our quivers
- **Example:** In G_3^2,

$$w_0 = \underbrace{\sigma_5}_{c_1^{rr}} \underbrace{\sigma_6\sigma_4\sigma_5}_{c_2^{r}} \underbrace{\sigma_1\sigma_2\sigma_3\sigma_6\sigma_4\sigma_5}_{c_3^{r}}$$



- Do these categorical group actions occur "in nature"?
- [Seidel-Thomas] Simplest geometric example of braid group action:
  - take cyclic subgroup $G < SL_2(\mathbb{C})$ acting on affine 2-space;
  - consider equivariant sheaves built from skyscraper sheaf of fixed point;
  - they are spherical objects in $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathrm{coh}\, \mathbb{A}^2)$ and their spherical twists satisfy braid relations.
- **Thm:** Similarly, G_s^d acts on $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathrm{coh}\, \mathbb{A}^{d+1})$ for abelian $G < SL_{d+1}(\mathbb{C})$.

Thanks for listening!